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NUMERICAL RECKONING OF FIXED POINTS FOR GENERALIZED $(\alpha, \beta)-$NONEXPANSIVE MAPPINGS IN HYPERBOLIC SPACES

Jaynendra Shrivas

Department of Mathematics,
Govt. V. Y. T. PG Autonomous College,
Durg - 491001, Chhattisgarh, INDIA

E-mail : jayshrivas95@gmail.com

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Abstract: This paper deals with the SRJ iteration process for approximating the fixed point of generalized $(\alpha, \beta)-$nonexpansive mappings in hyperbolic spaces. Furthermore, we establish a strong and Δ-converges theorem for generalized (α, β)-nonexpansive mapping in hyperbolic space. Finally, we present a numerical example to illustrate our main result and then display the efficiency of the proposed algorithm compared to different iterative algorithms in the literature. Our results obtained in this paper improve, extend and unify some related results in the literature.

Keywords and Phrases: Hyperbolic spaces, generalized $(\alpha, \beta)-$nonexpansive mapping, strong and Δ-convergence theorems.

2020 Mathematics Subject Classification: 47H10, 54H25, 54E50.

1. Introduction

We recall the following: Let G be a nonempty subset of a Banach space X and $\Phi\colon G \to G$ a self-mapping. A point $x \in X$ is said to be a fixed point of Φ if $\Phi x = x$.

Many researchers attracted in the direction of approximating the fixed points of nonexpansive mapping and its generalized form [3, 4, 9, 12, 14, 15, 18, 20, 21, 29] in a hyperbolic space.

Remember that a selfmap Φ on a subset G of a Banach space X is called nonexpansive if

$$||\Phi x - \Phi y|| \leq ||x - y|| \ \ \forall x, y \in G. \tag{1.1}$$

Suzuki [30] made a significant breakthrough in 2008 by introducing a weak notion of nonexpansive operators. It is worth noting that a selfmap Φ of a metric space subset G is said to satisfy Condition (C) (also known as Suzuki map) if for any $x, y \in G$, we have

$$\frac{1}{2}||x - \Phi x|| \leq ||x - y|| \implies ||\Phi x - \Phi y|| \leq ||x - y||. \tag{1.2}$$

Remark 1.1. *It is clear that every nonexpansive map is Suzuki nonexpansive. However, an example in* [30] *shows that there exists maps which are Suzuki nonexpansive but not nonexpansive.*

In 2011, Aoyama and Kohsaka [5] proposed the class of α-nonexpansive maps as follows:

A selfmap Φ on a subset G of a Banach space is said to satisfy α-nonexpansive maps if one can find a real number $\alpha \in [0, 1)$ for any $x, y \in G$, we have

$$||\Phi x - \Phi y||^2 \leq \alpha||x - \Phi y||^2 + \alpha||y - \Phi x||^2 + (1 - 2\alpha)||x - y||^2. \tag{1.3}$$

In 2017, Pant and Shukla [28] proposed the class of α-nonexpansive maps as follows:

A selfmap Φ on a subset G of a Banach space is said to satisfy generalized α-nonexpansive maps if one can find a real number $\alpha \in [0, 1)$ for any $x, y \in G$, we have

$$\begin{aligned} \frac{1}{2}||x - \Phi x|| \leq ||x - y|| \implies ||\Phi x - \Phi y|| &\leq \alpha||y - \Phi x|| + \alpha||x - \Phi y|| \\ &+ (1 - 2\alpha)||x - y||. \end{aligned} \tag{1.4}$$

Remark 1.2. *It is clear that every Suzuki nonexpansive map is generalized 0-nonexpansive. However, an example in* [28] *shows that there exist maps which are generalized α-nonexpansive but not Suzuki nonexpansive.*

In 2019, Pant and Pandey [27] proposed the class of Reich–Suzuki type nonexpansive maps as follows:

A selfmap Φ on a subset G of a Banach space is said to satisfy β-Reich-Suzuki type nonexpansive maps if one can find a real number $\beta \in [0, 1)$ for any $x, y \in G$, we have

$$\begin{aligned} \frac{1}{2}||x - \Phi x|| \leq ||x - y|| \implies ||\Phi x - \Phi y|| &\leq \beta||x - \Phi x|| + \beta||y - \Phi y|| \\ &+ (1 - 2\beta)||x - y||. \end{aligned} \tag{1.5}$$

Remark 1.3. *It is clear that every Suzuki nonexpansive map is 0-Reich–Suzuki type nonexpansive. However, an example in* [27] *shows that there exists maps which are β-Reich–Suzuki type nonexpansive but not Suzuki nonexpansive.*

Definition 1.4. [33] *A selfmap Φ on a subset G of a Banach space is said to be generalized (α, β)-nonexpansive, if there exists real number $\alpha, \beta \in \mathbb{R}^+$ satisfying $\alpha + \beta < 1$ such that, for all $x, y \in G$*

$$\frac{1}{2}||x - \Phi x|| \leq ||x - y|| \implies ||\Phi x - \Phi y|| \leq \alpha||x - \Phi y|| + \alpha||y - \Phi x|| + \beta||x - \Phi x|| + \beta||y - \Phi y|| + (1 - 2\alpha - 2\beta)||x - y||. \tag{1.6}$$

The following proposition gives many examples of generalized (α, β)-nonexpansive mpas.

Remark 1.5. *Let a selfmap Φ on a subset G of a Banach space. Then, the following hold:*

1. *If Φ is Suzuki nonexpansive, then Φ is generalized $(0, 0)$-nonexpansive.*
2. *If Φ is generalized α-nonexpansive, then Φ is generalized $(\alpha, 0)$- nonexpansive.*
3. *If Φ is β-Reich–Suzuki type nonexpansive, then Φ is generalized $(0, \beta)$- nonexpansive.*

Iterative techniques for finding fixed points are an important and active research area in nonlinear analysis with numerous applications in computers, applied economics, physics and many other applied sciences [1]. Because the Picard iteration $x_{n+1} = \Phi x_n$ does not always converge to a fixed point of a given nonexpansive operator, we will present here some other well-known processes that not only converge to a fixed point of a given nonexpansive operator but also have a higher rate of convergence than the Picard iteration. Let we assume E be a nonempty convex subset of a Banach space, $\alpha_n, \beta_n, \gamma_n \in (0, 1)$ and $\Phi\colon G \to G$ be a given operator.

Over the last few years many iterative processes have been obtained in different domains to approximate fixed points of various classes of mappings. Mann iteration [25], Ishikawa iteration [13], Thakur et al. [32] and Ullah et al. [33] are the few basic iteration processes.

Mann [25] described one of the earlier iteration processes as follows:

$$\begin{cases} x_1 \in G, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n \Phi x_n, \quad n \geq 1. \end{cases} \tag{1.7}$$

The Mann iteration can be seen as a subset of the Ishikawa iteration process, which was described by Ishikawa in [13] as follows:

$$\begin{cases} x_1 \in G, \\ y_n = (1-\beta_n)x_n + \beta_n \Phi x_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n \Phi y_n, \quad n \geq 1. \end{cases} \tag{1.8}$$

Agarwal et al. [2] is the slightly modified the Ishikawa iteration and defined as follows:

$$\begin{cases} x_1 \in G, \\ y_n = (1-\beta_n)x_n + \beta_n \Phi x_n, \\ x_{n+1} = (1-\alpha_n)\Phi x_n + \alpha_n \Phi y_n, \quad n \geq 1. \end{cases} \tag{1.9}$$

We can infer from [2] that the Agarwal iterative process is superior to the earlier processes, namely the Picard, Mann and Ishikawa iterative processes, by a significant margin.

In 2016, Thakur et al. [32] proposed the iterative process listed below:

$$\begin{cases} x_1 \in G, \\ z_n = (1-\beta_n)x_n + \beta_n \Phi x_n, \\ y_n = \Phi((1-\alpha_n)x_n + \alpha_n z_n), \\ x_{n+1} = \Phi y_n, \quad n \geq 1. \end{cases} \tag{1.10}$$

Thakur et al. [32] demonstrated that the sequence $\{x_n\}$ defined by the iterative process (1.10) converges (in certain circumstances) to a fixed point of a given Suzuki's map. Furthermore, they built a new example of Suzuki's mappings Φ and demonstrated that the iterative process (1.10) converges to a fixed point faster than earlier iterative processes proposed by Mann [25], Ishikawa [13], Noor [26], S-iteration [2] and Abbas [1].

In 2018, Ullah et al. [11] introduced a new iterative process, which they call it "K" iteration process, as follows:

$$\begin{cases} x_1 \in G, \\ z_n = (1-\beta_n)x_n + \beta_n \Phi x_n, \\ y_n = \Phi((1-\alpha_n)\Phi x_n + \alpha_n \Phi z_n), \\ x_{n+1} = \Phi y_n, \quad n \geq 1. \end{cases} \tag{1.11}$$

Question: Is it possible to develop an iteration process whose rate of convergence is even faster than the iteration processes defined above?

As a very straight forward answer, in 2023, Dashputre et al. [10] proposed the SRJ-iteration process, as follows:

Let G be a nonempty, closed and convex subset of a Banach space X and $\Phi\colon G \to G$ be a mapping. Let $x_1 \in G$ be arbitrary and the sequence $\{x_n\}$ generated iteratively by

$$\begin{cases} x_1 \in G, \\ z_n = \Phi((1-\alpha_n)x_n + \alpha_n \Phi x_n), \\ y_n = \Phi((1-\beta_n)z_n + \beta_n \Phi z_n), \\ x_{n+1} = \Phi((1-\gamma_n)y_n + \gamma_n \Phi y_n), \quad n \geq 1 \end{cases} \tag{1.12}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in (0,1).

2. Preliminaries

Throughout this paper, we consider the following definition of a hyperbolic space introduced by Kohlenbach [22].

Definition 2.1. *A metric space* (X, d) *is said to be a hyperbolic space if there exists a map* $W\colon X^2 \times [0,1] \to X$ *satisfying*
(i) $d(\rho, W(x, y, \alpha)) \leq \alpha d(\rho, x) + (1-\alpha)d(\rho, y)$,
(ii) $d(W(x, y, \alpha), W(x, y, \beta)) = |\alpha - \beta| d(x, y)$,
(iii) $W(x, y, \alpha) = W(y, x, (1-\alpha))$,
(iv) $d(W(x, z, \alpha), W(y, w, \alpha)) \leq \alpha d(x, y) + (1-\alpha)d(z, w)$,
for all $x, y, z, w \in X$ *and* $\alpha, \beta \in [0,1]$.

Definition 2.2. [31] *A metric space is said to be convex, if a triple* (X, d, W) *satisfy only (i) in Definition 2.1.*

Definition 2.3. [31] *A subset* G *of a hyperbolic space* X *is said to be convex, if* $W(x, y, \alpha) \in G$ *for all* $x, y \in G$ *and* $\alpha \in [0,1]$.

If $x, y \in X$ *and* $\lambda \in [0,1]$, *then we use the notation* $(1-\lambda)x \oplus \lambda y$ *for* $W(x, y, \lambda)$. *The following holds even for more general setting of convex metric space* [31] *: for all* $x, y \in X$ *and* $\lambda \in [0,1]$,

$$d(x, (1-\lambda)x \oplus \lambda y) = \lambda d(x, y)$$

and

$$d(y, (1-\lambda)x \oplus \lambda y) = (1-\lambda)d(x, y).$$

Thus

$$1x \oplus 0y = x, \;\; 0x \oplus 1y = y$$

and

$$(1-\lambda)x \oplus \lambda x = \lambda x \oplus (1-\lambda)x = x.$$

Definition 2.4. [23] *A hyperbolic space (X, ∂, W) is said to be uniformly convex, if for any $\rho, x, y \in X,\ r > 0$ and $\epsilon \in (0, 2]$, there exists a $\delta \in (0, 1]$ such that*

$$d\left(\frac{1}{2}x \oplus \frac{1}{2}y, \rho\right) \leq (1 - \delta)r,$$

whenever $d(x, \rho) \leq r,\ d(y, \rho) \leq r$ and $d(x, y) \geq \epsilon r$.

Definition 2.5. *A map $\eta\colon (0, \infty) \times (0, 2] \to (0, 1]$ which provides such a $\delta = \eta(r, \epsilon)$ for given $r > 0$ and $\epsilon \in (0, 2]$, is known as modulus of uniform convexity. We call η monotone if it decreases with r (for a fixed ϵ).*

In [23], Luestean proved that every CAT(0) space is a uniformly convex hyperbolic space with modulus of uniform convexity $\eta(r, \epsilon) = \frac{\epsilon^2}{8}$ quadratic in ϵ.

Now we give the concept of Δ-convergence and some of its basic properties.

Let G be a nonempty subset of metric space (X, d) and $\{y_n\}$ be any bounded sequence in X while $diam(G)$ denotes the diameter of G. Consider a continuous functional $r_a(., \{y_n\})\colon X \to R^+$ defined by

$$r_a(y, \{y_n\}) = \limsup_{n\to+\infty} d(y_n, y),\ y \in X.$$

The infimum of $r_a(., \{y_n\})$ over G is said to be an asymptotic radius of $\{y_n\}$ with respect to G and it is denoted by $r_a(G, \{y_n\})$. A point $z \in G$ is said to be an asymptotic center of the sequence $\{y_n\}$ with respect to G if

$$r_a(z, \{y_n\}) = inf\{r_a(y, \{y_n\})\colon y \in G\}.$$

The set of all asymptotic center of $\{y_n\}$ with respect to G is denoted by $AC(G, \{y_n\})$. The set $AC(G, \{y_n\})$ may be empty, singleton or have infinitely many points. If the asymptotic radius and asymptotic center are taken with respect to whole space X, then they are denoted by $r_a(X, \{y_n\}) = r_a(\{y_n\})$ and $AC(X, \{y_n\}) = AC(\{y_n\})$, respectively. We know that for $y \in X,\ r_a(y, \{y_n\}) = 0$ if and only if $\lim_{n\to+\infty} y_n = y$ and every bounded sequence has a unique asymptotic center with respect to closed convex subset in uniformly convex Banach spaces.

Definition 2.6. *The sequence $\{y_n\}$ in X is said to be Δ-convergent to $y \in X$, if y is unique asymptotic center of the every subsequence $\{u_n\}$ of $\{y_n\}$. In this case, we write $\Delta - \lim_{n\to\infty} y_n = y$ and call y is the Δ-limit of $\{y_n\}$.*

Lemma 2.7. [24] *Let (X, d, W) be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Then every bounded sequence $\{x_n\}$ in X has a unique asymptotic center with respect to any nonempty closed convex subset G of X.*

Consider the following lemma of Khan et al. [17] which we use in the sequel.

Lemma 2.8. *Let* (X, d, W) *be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity* η*. Let* $x \in X$ *and* $\{t_n\}$ *be a sequence in* $[a, b]$ *for some* $a, b \in (0, 1)$*. If* $\{x_n\}$ *and* $\{y_n\}$ *are sequences in* X *such that*

$$\limsup_{n\to+\infty} d(x_n, x) \leq c,$$

$$\limsup_{n\to+\infty} d(y_n, x) \leq c$$

and

$$\limsup_{n\to+\infty} d(W(x_n, y_n, t_n), x) = c$$

for some $c \geq 0$*, then* $\lim_{n\to+\infty} d(x_n, y_n) = 0$.

Definition 2.9. *Let* G *be a nonempty convex closed subset of a hyperbolic space* X *and* $\{x_n\}$ *be a sequence in* X *. Then* $\{x_n\}$ *is said to be Fejér monotone with respect to* M *if for all* $x \in G$ *and* $n \in \mathbb{N}$*,*

$$d(x_{n+1}, x) \leq d(x_n, x).$$

Assume that G is a nonempty subset of a hyperbolic space (X, d) and $\Phi\colon G \to G$ is a mapping and $F(\Phi) = t \in G\colon \Phi t = t$ is the set of all fixed points of the map Φ. The mapping $\Phi\colon G \to G$ is called nonexpansive, if $||\Phi t - \Phi\rho|| \leq ||t - \rho||$ for all $t, \rho \in G$ and is called quasi-nonexpansive, if $F(\Phi) \neq \emptyset$ and $||\Phi t - q|| \leq ||t - q||$ for all $t \in G$ and $q \in F(\Phi)$.

We can easily prove the following Proposition.

Proposition 2.10. *Let* $\{x_n\}$ *be a sequence in* X *and* G *be a nonempty subset of* X*. Let* $\Phi\colon G \to G$ *be a nonexpansive mapping with* $F(\Phi) \neq \emptyset$*. Suppose that* $\{x_n\}$ *is Fejér monotone with respect to* G*. Then we have the followings:*
(1) $\{x_n\}$ *is bounded.*
(2) The sequence $\{d(x_n, p)\}$ *is decreasing and converges for all* $p \in F(\Phi)$*.*
(3) $\lim_{n\to+\infty} D(x_n, F(\Phi))$ *exists, where* $D(x, A) = \inf_{y\in A} d(x, y)$*.*

Lemma 2.11. [33] *Assume that* G *is a nonempty subset of a hyperbolic space* X *and* $\Phi\colon G \to G$ *is generalized* (α, β)*-nonexpansive. Then for* $x, y \in G$*,*

1. $||\Phi x - \Phi^2 x|| \leq ||x - \Phi x||$.
2. Either $\frac{1}{2}||x - \Phi x|| \leq ||x - y||$ *or* $\frac{1}{2}||\Phi x - \Phi^2 x|| \leq ||\Phi x - y||$.

3. Either
$||\Phi x-\Phi y|| \leq \alpha||x-\Phi y||+\alpha||y,-\Phi x||+\beta||x-\Phi x||+\beta||y-\Phi y||+(1-2\alpha-2\beta)||x-y||$
or
$||\Phi^2 x-\Phi y|| \leq \alpha||\Phi x-\Phi y||+\alpha||y-\Phi^2 x||+\beta||\Phi x-\Phi^2 x||+\beta||y-\Phi y||+(1-2\alpha-2\beta)||\Phi x-y||$.

Lemma 2.12. [33] *Assume that G is a nonempty subset of a hyperbolic space X and $\Phi\colon G \to G$ is generalized (α,β)-nonexpansive. Then for $x,\rho \in G$ with $x \leq \rho$,*

$$||x-\Phi x|| \leq \left(\frac{3+\alpha+\beta}{1-\alpha-\beta}\right)||x-\Phi x||+||x-\rho||.$$

Definition 2.13. *Assume that G is a nonempty subset of a hyperbolic space X and $\Phi\colon G \to G$ is a generalized (α,β)-nonexpansive mapping with $F(\Phi) \neq \emptyset$. Then Φ is quasi-nonexpansive.*

Lemma 2.14. [29] *Let X be complete uniformly convex hyperbolic space with monotone modulus of convexity η, G be a nonempty closed convex subset of X and $\Phi\colon G \to G$ be a generalized (α,β)-nonexpansive mapping. If $\{x_n\}$ is a bounded sequence in G such that $\lim_{n\to+\infty} d(x_n,\Phi x_n)=0$, then Φ has a fixed point in G.*

Lemma 2.15. [29] *Let G be a nonempty, bounded, closed and convex subset of a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η and Φ be a generalized (α,β)-nonexpansive mapping on G. Suppose that $\{x_n\}$ is a sequence in G, with $d(x_n,\Phi x_n) \to 0$. If $AC(G,\{x_n\})=\rho$, then ρ is a fixed point of Φ. Moreover, $F(\Phi)$ is closed and convex.*

3. Main Result

Now, we establish the convergence results for SRJ-iteration process for generalized (α,β)-nonexpansive mappings in hyperbolic spaces, as follows: Let G be a nonempty, closed and convex subset of a hyperbolic space X and Φ be a generalized (α,β)-nonexpansive mapping on G. For any $x_1 \in G$ the sequence $\{x_n\}$ is defined by

$$\begin{cases} z_n = W(\Phi\sigma_n,0,0), \\ \sigma_n = W(x_n,\Phi x_n,\alpha_n), \\ y_n = W(\Phi\nu_n,0,0), \\ \nu_n = W(z_n,\Phi z_n,\beta_n), \\ x_{n+1} = W(\Phi\varrho_n,0,0), \\ \varrho_n = W(y_n,\Phi y_n,\gamma_n),\ \forall n \in \mathbb{N} \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in (0,1). This section establishes some significant strong and Δ-convergence results for operators with generalized (α, β)-nonexpansive mapping. Our results will generalize the results of Ullah et al. [33] and Dashputre et al [10].

Theorem 3.1. *Let Gbe a nonempty, closed and convex subset of a hyperbolic space X and $\Phi\colon G \to G$ be a generalized (α, β)-nonexpansive mapping. If $\{x_n\}$ is a sequence defined by (3.1), then $\{x_n\}$ is Fejér monotone with respect to $F(\Phi)$.*

Proof. Since Φ is a generalized (α, β)-nonexpansive, for $\rho \in F(\Phi)$, we have

$$\frac{1}{2}d(\rho, \Phi\rho) = 0 \leq d(\rho, x_n),$$

$$\frac{1}{2}d(\rho, \Phi\rho) = 0 \leq d(\rho, y_n)$$

and

$$\frac{1}{2}d(\rho, \Phi\rho) = 0 \leq d(\rho, z_n),$$

for all $n \in \mathbb{N}$. Now, also we have

$$\begin{aligned} d(\Phi\rho, \Phi x_n) \leq\ & \alpha d(\rho, \Phi x_n) + \alpha d(x_n, \Phi\rho) + \beta d(\rho, \Phi\rho) + \beta d(x_n, \Phi x_n) \\ & + (1 - 2\alpha - 2\beta) d(\rho, x_n), \\ d(\Phi\rho, \Phi y_n) \leq\ & \alpha d(\rho, \Phi y_n) + \alpha d(y_n, \Phi\rho) + \beta d(\rho, \Phi\rho) + \beta d(y_n, \Phi y_n) \\ & + (1 - 2\alpha - 2\beta) d(\rho, y_n) \end{aligned}$$

and

$$\begin{aligned} d(\Phi\rho, \Phi z_n) \leq\ & \alpha d(\rho, \Phi z_n) + \alpha d(z_n, \Phi\rho) + \beta d(\rho, \Phi\rho) + \beta d(z_n, \Phi z_n) \\ & + (1 - 2\alpha - 2\beta) d(\rho, z_n). \end{aligned}$$

Now, using (3.1) and Definition 2.13,

$$d(\Phi\rho, \Phi x_n) \leq d(\rho, x_n),$$

$$d(\Phi\rho, \Phi y_n) \leq d(\rho, y_n)$$

and

$$d(\Phi\rho, \Phi z_n) \leq d(\rho, z_n). \tag{3.2}$$

Using Definition 2.13 and (3.1), we get

$$\begin{aligned} d(z_n, p) &= d(W(\Phi\sigma_n, 0, 0)), \rho) \\ &= d(\Phi\sigma_n, \rho) \\ &\leq d(\sigma_n, \rho) \\ &= d(W(x_n, \Phi x_n, \alpha_n), \rho) \\ &\leq ((1-\alpha_n))d(x_n, \rho) + \alpha_n d(\Phi x_n, \rho) \\ &\leq (1-\alpha_n)d(x_n, \rho) + \alpha_n d(x_n, \rho) \\ &\leq d(x_n, \rho). \end{aligned} \tag{3.3}$$

Using Definition 2.13, (3.1) and (3.3), we get

$$\begin{aligned} d(y_n, p) &= d(W(\Phi\nu_n, 0, 0)), \rho) \\ &= d(\Phi\nu_n, \rho) \\ &\leq d(\nu_n, \rho) \\ &= d(W(z_n, \Phi z_n, \beta_n), \rho) \\ &\leq (1-\beta_n)d(z_n, \rho) + \beta_n d(\Phi z_n, \rho) \\ &\leq (1-\beta_n)d(z_n, \rho) + \beta_n d(z_n, \rho) \\ &\leq (1-\beta_n)d(x_n, \rho) + \beta_n d(x_n, \rho) \\ &\leq d(x_n, \rho). \end{aligned} \tag{3.4}$$

Using Definition 2.13, (3.1), (3.3) and (3.4), we get

$$\begin{aligned} d(x_{n+1}, p) &= d(W(\Phi\varrho_n, 0, 0)), \rho) \\ &= d(\Phi\varrho_n, \rho) \\ &\leq d(\varrho_n, \rho) \\ &= d(W(y_n, \Phi y_n, \gamma_n), \rho) \\ &\leq (1-\gamma_n)d(y_n, \rho) + \gamma_n d(\Phi y_n, \rho) \\ &\leq (1-\gamma_n)d(y_n, \rho) + \gamma_n d(y_n, \rho) \\ &\leq (1-\gamma_n)d(x_n, \rho) + \gamma_n d(x_n, \rho) \\ &\leq d(x_n, \rho). \end{aligned} \tag{3.5}$$

Hence, $\{x_n\}$ is Fejér monotone with respect to $F(\Phi)$.

Theorem 3.2. *Let G be a nonempty, closed and convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity*

η and Φ be a generalized (α,β)-nonexpansive mapping on G. If $\{x_n\}$ is a sequence defined by (3.1), then $F(\Phi)$ is nonempty if and only if the sequence $\{x_n\}$ is bounded and $\lim_{n\to+\infty} d(x_n,\Phi x_n)=0$.

Proof. Assume that $F(\Phi)$ is nonempty and let $\rho\in F(\Phi)$. From Theorem 3.1 and Proposition 2.10, we have $\{x_n\}$ is Fejér monotone with respect to $F(\Phi)$ and bounded such that $\lim_{n\to+\infty} D((x_n,F(\Phi))$ exists, let $\lim_{n\to+\infty} d(x_n,\rho)=l$.

Case I. Let $l=0$. Then

$$d(x_n,\Phi x_n)\leq d(x_n,\rho)+d(\rho,\Phi x_n),$$

from Definition 2.13,

$$d(x_n,\Phi x_n)\leq 2d(x_n,\rho).$$

On taking limit as $n\to+\infty$ both sides of the inequality,

$$\lim_{n\to+\infty} d(x_n,\Phi x_n)=0.$$

Case II. Let $l>0$. Then, since G is a generalized (α,β)-nonexpansive mapping, by Definition 2.13, for $\rho\in F(\Phi)$,

$$d(\Phi x_n,\rho)\leq d(x_n,\rho).$$

On taking lim sup as $n\to+\infty$ both sides of the inequality,

$$\limsup_{n\to+\infty} d(\Phi x_n,\rho)\leq l.$$

On taking lim sup as $n\to+\infty$ both sides of the (3.4),

$$\limsup_{n\to+\infty} d(z_n,\rho)\leq l. \tag{3.6}$$

From (3.5),

$$\begin{aligned} d(x_{n+1},p) &= d(W(\Phi\varrho_n,0,0)),\rho) \\ &= d(\Phi\varrho_n,\rho) \\ &\leq d(\varrho_n,\rho) \\ &= d(W(y_n,\Phi y_n,\gamma_n),\rho) \\ &\leq (1-\gamma_n)d(y_n,\rho)+\gamma_n d(\Phi y_n,\rho) \\ &\leq (1-\gamma_n)d(x_n,\rho)+\gamma_n d(y_n,\rho). \end{aligned}$$

It follows that

$$\begin{aligned} d(x_{n+1},\rho) - d(x_n,\rho) &\leq \gamma_n(d(y_n,\rho) - d(x_n,\rho)) \\ d(x_{n+1},\rho) - d(x_n,\rho) &\leq \frac{d(x_{n+1},\rho) - d(x_n,\rho)}{\gamma_n} \\ &\leq d(y_n,\rho) - d(x_n,\rho) \end{aligned}$$

$$d(x_{n+1},\rho) \leq d(y_n,\rho).$$

On taking lim sup as $n \to +\infty$ both sides of the inequality,

$$l \leq \liminf_{n\to+\infty} d(y_n,\rho). \tag{3.7}$$

From (3.6) and (3.7),

$$\lim_{n\to+\infty} d(y_n,\rho) = l.$$

On taking lim sup as $n \to +\infty$ in (3.3),

$$\limsup_{n\to+\infty} d(z_n,\rho) \leq l. \tag{3.8}$$

From (3.5),

$$\begin{aligned} d(x_{n+1},p) &= d(W(\Phi\varrho_n,0,0)),\rho) \\ &= d(\Phi\varrho_n,\rho) \\ &\leq d(\varrho_n,\rho) \\ &= d(W(y_n,\Phi y_n,\gamma_n),\rho) \\ &\leq (1-\gamma_n)d(y_n,\rho) + \gamma_n d(\Phi y_n,\rho) \\ &\leq (1-\gamma_n)d(z_n,\rho) + \gamma_n d(y_n,\rho) \\ &\leq (1-\gamma_n)d(z_n,\rho) + \gamma_n d(z_n,\rho) \\ &\leq d(z_n,\rho). \end{aligned}$$

On taking lim inf as $n \to +\infty$ both sides of the inequality,

$$l \leq \liminf_{n\to+\infty} d(z_n,\rho). \tag{3.9}$$

From (3.8) and (3.9),

$$\lim_{n\to+\infty} d(z_n,\rho) = l.$$

Therefore, by (3.3)

$$\begin{aligned} l &= \limsup_{n\to+\infty} d(z_n, \rho) \\ &\le \limsup_{n\to+\infty} d(W(x_n, \Phi x_n, \alpha_n), \rho) \\ &\le \limsup_{n\to+\infty} [(1-\alpha_n)d(x_n, \rho) + \alpha_n d(\Phi x_n, \rho)] \\ &\le \limsup_{n\to+\infty} [(1-\alpha_n)d(x_n, \rho) + \alpha_n d(x_n, \rho)] \\ &\le \limsup_{n\to+\infty} d(x_n, \rho) = l. \end{aligned}$$

By Lemma 2.8, $\lim_{n\to+\infty} d(x_n, \Phi x_n) = 0$.

Conversely, assume that $\{x_n\}$ is bounded and $\lim_{n\to+\infty} d(x_n, \Phi x_n) = 0$. Then, from Lemma 2.14, we have $\Phi\rho = \rho$, that is, $F(\Phi)$ is nonempty.

Theorem 3.3. *Let G be a nonempty, closed and convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η. Let $\Phi\colon G \to G$ be a generalized (α, β)-nonexpansive mapping with $F(\Phi) \neq \emptyset$. Then the sequence $\{x_n\}$ defined in (3.1), is Δ-convergent to a fixed point of Φ.*

Proof. From Theorem 3.1, we observe that $\{x_n\}$ is a bounded sequence, therefore $\{x_n\}$ has a Δ-convergent subsequence. Now we will prove that every Δ-convergent subsequence of $\{x_n\}$ has a unique $\Delta - limit$ in $F(\Phi)$. For this, let y and z be $\Delta - limit$ of the subsequences $\{y_n\}$ and $\{z_n\}$ of $\{x_n\}$ respectively.

Now by Lemma 2.7, $AC(G, \{y_n\}) = \{y_n\}$ and $AC(G, \{z_n\}) = \{z_n\}$. By Theorem 3.2, we have $\lim_{n\to+\infty} d(y_n, \Phi y_n) = 0$.

Now we will prove that y and z are fixed points of Φ and they are same. If not, then by the uniqueness of the asymptotic center

$$\begin{aligned} \limsup_{n\to+\infty} d(x_n, y) &= \limsup_{n\to+\infty} d(y_n, y) \\ &< \limsup_{n\to+\infty} d(y_n, z) \\ &= \limsup_{n\to+\infty} d(x_n, z) \\ &= \limsup_{n\to+\infty} d(z_n, z) \\ &< \limsup_{n\to+\infty} d(z_n, y) \\ &= \limsup_{n\to+\infty} d(x_n, y) \end{aligned}$$

which is a contradiction. Hence $y = z$ and sequence $\{x_n\}$ is Δ-convergent to a unique fixed point of Φ.

Theorem 3.4. *Let G be a nonempty, closed and convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and $\Phi\colon G \to G$ be a generalized (α, β)-nonexpansive mapping with $F(\Phi) \neq \emptyset$. Then the sequence $\{x_n\}$ which is defined by (3.1), converges strongly to some fixed point of Φ if and only if $\liminf_{n\to+\infty} D(x_n, F(\Phi)) = 0$, where $D(x_n, F(\Phi)) = \inf_{y\in F(\Phi)} d(x_n, y)$.*

Proof. Assume that $\{x_n\}$ converges strongly to $y \in F(\Phi)$. Therefore we have $\lim_{n\to+\infty} d(x_n, y) = 0$. Since $0 \leq D(x_n, F(\Phi)) \leq d(x_n, y)$, we have

$$\liminf_{n\to+\infty} D(x_n, F(\Phi)) = 0.$$

Next, we prove sufficient part. From Lemma 2.15, the fixed point set $F(\Phi)$ is closed. Suppose that

$$\liminf_{n\to+\infty} D(x_n, F(\Phi)) = 0.$$

Then, from (3.5), we have

$$D(x_{n+1}, F(\Phi)) \leq D(x_n, F(\Phi)).$$

From Theorem 3.1 and Proposition 2.10, we have $\lim_{n\to+\infty} d(x_n, F(\Phi))$ exists. Hence

$$\lim_{n\to+\infty} D(x_n, F(\Phi)) = 0.$$

Consider the subsequence $\{x_{n_k}\}$of $\{x_n\}$ such that $d(x_{n_k}, p_k) < \frac{1}{2^k}$ for all $k \geq 1$, where $\{p_k\}$ is in $F(\Phi)$. From (3.4), we have

$$d(x_{n_{k+1}}, p_k) \leq d(x_{n_k}, p_k) < \frac{1}{2^k},$$

which implies that

$$\begin{aligned} d(p_{k+1}, p_k) &\leq d(p_{k+1}, x_{n_{k+1}}) + d(x_{n_{k+1}}, p_k) \\ &< \frac{1}{2^{k+1}} + \frac{1}{2^k} \\ &< \frac{1}{2^{k-1}}. \end{aligned}$$

This shows that $\{p_k\}$ is a Cauchy sequence. Since $F(\Phi)$is closed, $\{p_k\}$ is a convergent sequence. Let $\lim_{k\to\infty} p_k = p$. Then we know that $\{x_n\}$ converges to y. Since

$$d(x_{n_k}, y) \leq d(x_{n_k}, p_k) + d(p_k, y),$$

we have

$$\lim_{k\to\infty} d(x_{n_k}, y) = 0.$$

Since $\lim_{n\to+\infty} d(x_n, y)$ exists, the sequence $\{x_n\}$ converges to y.

Recall that a mapping Φ from a subset of a hyperbolic space X into itself with $F(\Phi) \neq \emptyset$ is said to satisfy condition (I) if there exists a nondecreasing function $f\colon [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(t) > 0$ for $t \in (0,\infty)$ such that

$$d(x, \Phi x) \geq f(D(x, F(\Phi))),$$

for all $x \in G$.

Theorem 3.5. *Let G be a nonempty, closed and convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and $\Phi\colon G \to G$ be a generalized (α,β)-nonexpansive mapping. Moreover, Φ satisfies the condition (I) with $F(\Phi) \neq \emptyset$. Then the sequence $\{x_n\}$ which is defined by (3.1), converges strongly to some fixed point of Φ.*

Proof. From Lemma 2.15, we have $F(\Phi)$ is closed. Observe that by Theorem 3.2, we have $\lim_{n\to+\infty} d(x_n, \Phi x_n) = 0$. It follows from the condition (I) that

$$\lim_{n\to+\infty} f(D(x_n, F(\Phi))) \leq \lim_{n\to+\infty} d(x_n, \Phi x_n).$$

Thus, we get $\lim_{n\to+\infty} f(D(x_n, F(\Phi))) = 0$. Since $f\colon [0,1) \to [0,1)$ is a nondecreasing mapping with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$, we have $\lim_{n\to+\infty} D(x_n, F(\Phi)) = 0$. Rest of the proof follows in lines of Theorem 3.4. Hence the sequence $\{x_n\}$ is convergent to $p \in F(\Phi)$. This completes the proof.

4. Numerical Example

The following example shows that there exist maps which are generalized (α,β)-nonexpansive but neither generalized α-nonexpansive nor β-Reich–Suzuki type.

Example 4.1. [33] Let $K = \mathbb{R}^+$ with usual norm $||.||$. Then map $\Phi\colon K \to G$ is defined as

$$\Phi x = \begin{cases} 0, & if \quad x \in [0, \frac{1}{2}], \\ \frac{x}{2}, & if \quad x \in (\frac{1}{2}, \infty), \end{cases}$$

for all $x \in K$. Then, it is easy to see that Φ is neither generalized $\frac{1}{4}$-nonexpansive nor $\frac{1}{4}$-Reich-Suzuki type. Hence, Φ satisfies the generalized $(\frac{1}{4}, \frac{1}{4})$-nonexpansive. We obtained the influence of initial point for the SRJ iteration algorithm (1.12) by $\alpha_n = 0.90, \beta_n = 0.65, \gamma_n = 0.85$ and $x_1 = 1000$.

Table 1: Convergence of SRJ iteration for fixed point 0.

No. of iteration	Ishikawa iteration	Agrawal iteration	Thakur iteration	K iteration	SRJ iteration
0	1000	1000	1000	1000	1000
1	403.7500000	353.7500000	176.8750000	100.9375000	26.68359375
2	163.0140625	125.1390625	31.28476563	10.18837891	0.712014175
3	65.81692773	44.26794336	5.533492920	1.028389496	0
4	26.57358457	15.65978496	0.978736560	0	0
5	10.72908477	5.539648931	0	0	0
6	4.331867976	1.959650809	0	0	0
7	1.748991695	0.693226474	0	0	0
8	0.706155397	0.034661324	0	0	0
9	0.007061842	0	0	0	0
10	0.007061554	0	0	0	0

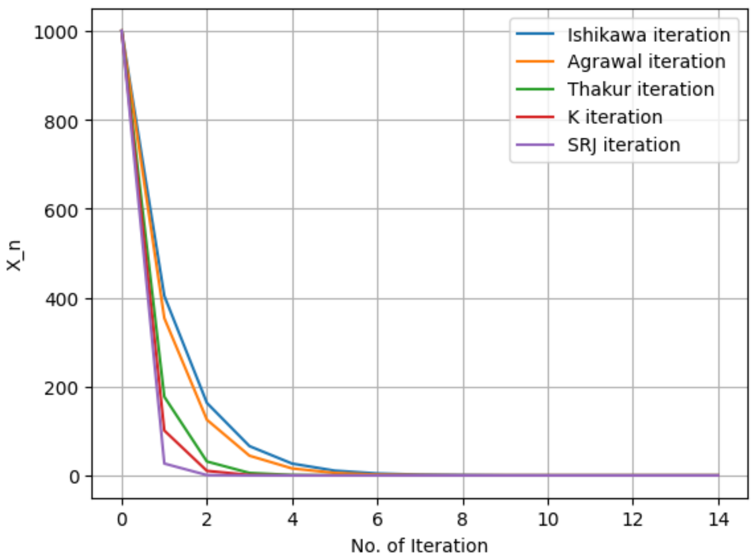

Figure 1: Convergence of Ishikawa, Agrawal, Thakur, K and SRJ iterations

5. Conclusion

In this work, we present some fixed point results for generalized (α, β)- nonexpansive mappings and also use an SRJ iterative algorithm for approximating the fixed point of this class of mappings in the framework of hyperbolic spaces. We have also performed some numerical computations to validate the claims and results of the paper. Our numerical experiment shows that our iterative algorithm is better than some existing iterative algorithms in the literature.

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